

AD-A048 649

FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/O 12/1
PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN THE AVERAGE EX--ETC(U)
JUL 77 Y Y REMEZ
FTD-ID(RS)T-1265-77 NL

UNCLASSIFIED

NL

| OF |
AD
A048649

END
DATE
FILMED
2 -78
DDC

AD-A048649

FTD-ID(RS)T-1265-77

FOREIGN TECHNOLOGY DIVISION

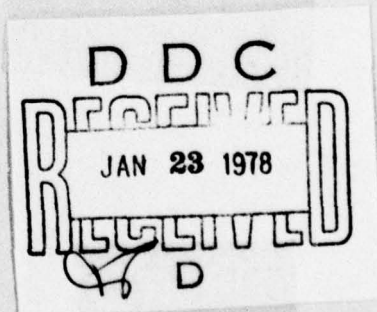
①



PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN
THE AVERAGE EXPONENTIAL AND CHEBYSHEV APPROXIMATIONS
(REPORT II)

by

Ye. Ya. Remez



Approved for public release;
distribution unlimited.

ACCESSION for	
NTIS	White Section <input checked="" type="checkbox"/>
DDI	Self Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION	
BY	
DISTRIBUTION/AVAILABILITY CODES	
Dist.	AVAIL. AND/OR SPECIAL
A	

FTD-

ID(RS)T-1265-77

UNEDITED MACHINE TRANSLATION

FTD-ID(RS)T-1265-77

29 July 1977

MICROFICHE NR: *FD-77-C-000910*

PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN
THE AVERAGE EXPONENTIAL AND CHEBYSHEV APPROXIMATIONS
(REPORT II)

By: Ye. Ya. Remez

English pages: 26

Source: Sbornik Trudov Instituta Matematiki
Akademii Nauk USSR, Pt. II, No. II, 1948
PP. 24-35

Country of origin: USSR

This document is a machine translation

Requester: AFFDL/FBRD

Approved for public release; distribution unlimited

THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION.

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

FTD-

ID(RS)T-1265-77

Date 29 July 19 77

Page 24.

PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN THE AVERAGE
EXPONENTIAL AND CHEBYSHEV APPROXIMATIONS (REPORT II *).

~~Ye.~~ Ya. Remez.

FOOTNOTE *. The first part of the work is printed in Ukrainian language in the previous issue of this collection. See also three author's compressed report/communications on these questions in the PAS of the USSR (1947), of Vol. LVIII.

The being encountered in this article literary references, marked by numerals in brackets, are related to the list of the cited literature of the first part of the work. ENDFOOTNOTE.

§13. Subsequently we are intended to place in parallel with the results, obtained in §§3-11 the previous report/communication, the analogous facts, which concern the processes interpolational, connected with construction for the assigned on segment $[a, b]$, set

of functions $\{v(x)\}$ the solutions of the problem of the average exponential approach/approximation on the alternating/variable finite set N of the points of segment $[a, b]$, when a number of points consecutively and unlimitedly grows.

Let us begin from the establishment of some basic common/general/total facts.

In the same way as in §3 and the following after it, let us be hearth $v_0(x), v_1(x), \dots, v_n(x)$ to understand certain assigned and fixed/recorded system real, continuous on the segment $[a, b]$ ($b - a = \tau$) and of linearly independent on it functions, and from the common/general/total set of polynomials $\Omega(x) = \sum_{i=0}^n c_i v_i(x)$ let us isolate as the permissible polynomials those, for which $c_0 = 1$, i.e.,

$$\Phi(x) = v_0(x) + \sum_{i=1}^n c_i v_i(x).$$

Let us place in the correspondence to each positive integer value of number $N \gg n + 1$ certain specific totally disconnected set $E_N \subset [a, b]$ consisting of N of points ³³:

$$x_{N1}, x_{N2}, \dots, x_{NN} \subset [a, b]. \quad (134)$$

FOOTNOTE ³³. Essence of further examinations did not change if we gave N not all positive integer values $\gg n + 1$, but certain

infinite-increasing particular sequence of the same. ENDFOOTNOTE.

Analogous with problems (9), (10) they are formulated for each of the final totally disconnected sets E_N in question the problem:

$$\delta_{m,N}[\Phi] = \delta_{m,N}(c_1, \dots, c_n) = \left[\frac{1}{N} \sum_{i=1}^N |\Phi(x_{Ni})|^m \right]^{\frac{1}{m}} = \min \quad (m > 1) \quad (135)$$

and

$$\delta_{0N}[\Phi] = \delta_{0N}(c_1, \dots, c_n) = \max_{x \in E_N} |\Phi(x)| = \min, \quad (136)$$

solutions of which (for the second task - not necessarily the only ones) let us designate respectively by $\Phi_{m,N}(x)$ and $\Phi_{0N}(x)$.

Page 25.

We will assume still:

$$\delta_{0N}[\Phi_{0N}] = \varphi_N, \quad \delta_{0N}[\Phi_{m,N}] = \varphi_N(1 + 2\beta_{m,N}) \quad (137)$$

and, on the other hand, preserving designations (10), (13),

$$\delta_0[\Phi_{m,N}] = \varphi + \beta_{m,N} = \varphi(1 + 2\beta_{m,N}). \quad (138)$$

The establishment of the corresponding estimate/evaluations for value $\beta_{m,N}$ at the large values of m and N will compose the final goal of further examinations (§15).

In order to ensure the uniqueness of the solution of problem (135) for each ³⁴ value of N , it is sufficient (comp. §1) to agree, during the composition of totally disconnected sets $E_N (N=n+1, n+2, \dots)$, each time to attend to to include/connect in its composition any and the points of segment $[a, b]$, on which the functions $v_1(x), \dots, v_n(x)$ are linearly independent, which is surely possible ³⁵ in view of the assumed linear independence of functions $v_i(x)$ from segment $[a, b]$.

FOOTNOTES ³⁴. For sufficiently large values of N the linear independence of functions $v_i(x)$ for E_N [including even $v_0(x)$] and, therefore, the uniqueness of the solution of problem (135) in actuality turn out to be automatically provided for under condition (139) already on the strength of the assumed linear independence of continuous functions $v_i(x)$ on segment $[a, b]$ as of this it is possible to be convinced by the reasoning, analogous to given below in the proof of theorem (comp. also observation to theorem).

³⁵. Comp. chapter VIII my monograph [10]. ENDFOOTNOTES.

Very significant for future reference it turns out to be,

furthermore, the requirement in order that during growth in number N of the point of set E_N they would be distributed over segment $[a, b]$ at least approximate-evenly. This requirement we can understand here in the following sufficiently soft sense:

for any fixed/recorded particular segment $[\alpha, \beta] \subset [a, b]$ must exist this positive number x , independent of N , but which is powerful of depending on the segment itself $[\alpha, \beta]$ in order that designate through $\mathfrak{N}_{\alpha, \beta, N}$ a number of points of set E_N , which fall on this particular segment $[\alpha, \beta]$ would satisfy the inequality:

$$\frac{\mathfrak{N}_{\alpha, \beta, N}}{N} > x \frac{\beta - \alpha}{b - a} \text{ for sufficiently large } N. \quad (139)$$

By assuming these conditions made, we now let establish/install the following, that has fundamental value for further conclusion/derivations the theorem, analogous by that, that was establish/installed in §1 for the in principle simpler case of the fixed/recorded multitude E .

Theorem. The coefficients $c_{m, N, i}$ the polynomial

$$\Phi_{m, N}(x) = v_m(x) + \sum_{i=1}^n c_{m, N, i} v_i(x), \quad (140)$$

employee by the solution of problem (135), are uniformly bounded in their set for all values of $N \geq n + 1$ and $m > 1$ in question.

Page 26.

Proof. For any that which was fix/recorded E_N and, therefore, generally, for limited $N \leq \bar{N}$ the limitedness of coefficients $c_{n,N,i}$ escape/ensues from theorem of §1. Therefore, proving this theorem of the contrary, it suffices to assume existence of the sequence of the permissible polynomials - the solutions of the corresponding problem of average exponential approach/approximation (135):

$$\Phi_1 = \Phi_{n_1, N_1}(x), \Phi_2 = \Phi_{n_2, N_2}(x), \dots, \Phi_{(v)} = \Phi_{n_v, N_v}(x), \dots, \quad (141)$$

for which simultaneously they would take place of the relationship:

$$N_v \rightarrow \infty, L_v = L[\Phi_{(v)}] \rightarrow \infty \text{ with } v \rightarrow \infty, \quad (142)$$

where L it designates value (3).

Let us consider (generally, not belonging already to the class "permissible") the "given polynomials":

$$\Omega_v(x) = \Omega_v = \frac{\Phi_{(v)}}{L_v} \quad (v=1, 2, 3, \dots), \quad (143)$$

satisfying, obviously, the condition

$$L[\Omega_v] = 1 \quad (v=1, 2, 3, \dots). \quad (144)$$

Let it will be

$$\delta_{n_v, N_v}[\Phi_{(v)}] = \left[\frac{1}{N_v} \sum_{j=1}^{N_v} |\Phi_{(v)}(x_{N_v, j})|^2 \right]^{\frac{1}{2}} = M_v \quad (v=1, 2, 3, \dots). \quad (145)$$

From the requirement

$$M < \delta_{n_n, n_n}[\Phi_0] < \epsilon \quad (146)$$

it follows immediately:

$$\left[\frac{1}{N} \sum_{j=1}^{N_s} |\Omega_s(x_{nj})|^{1/n_s} \right]^{1/n_s} = \frac{M_s}{L_s} < \frac{\epsilon}{L_s} \rightarrow 0 \text{ with } n \rightarrow \infty. \quad (147)$$

On the other hand, as a result of the limitedness of the coefficients of the given polynomials Ω_s , expressed by condition (144), there is a particular sequence of indices n_s ($s = 1, 2, 3, \dots$), for which all $n + 1$ coefficients of polynomial Ω_s simultaneously approach the limiting values, which determine certain polynomial $\Omega_s(x)$. The latter satisfies, obviously, the same condition (144) (has a coefficient with $\Omega_s(x)$ the knowingly equal to zero).

Page 27.

Restoring, for larger simplicity, a designation n , instead of n_s , for the index of this particular sequence, we have, therefore,

$$\lim_{n \rightarrow \infty} \Omega_n(x) = \Omega_0(x) \text{ evenly for } a < x < b, \quad (148)$$

$$L(\Omega_0) = 1. \quad (149)$$

It is obvious with the fact that relationship (147) remains valid and for this particular sequence. Here, however, immediately is reveal/detected contradiction.

Actually, from (149), taking into account the linear independence of functions $\Omega_n(x)$ on $[a, b]$, we include that the polynomial $\Omega_0(x)$ is different of identical zero on $[a, b]$, and its module/modulus, being continuous function of x , it has certain positive minimum 2μ of any particular segment $[\alpha, \beta] \subset [a, b]$ length $\beta - \alpha = \lambda$. On the same segment $[\alpha, \beta]$ the given polynomials Ω_n at sufficiently large values $n > \bar{n}$ satisfy the condition

$$\min_{x \in [\alpha, \beta]} |\Omega_n(x)| > \mu \quad \text{for } n > \bar{n}. \quad (150)$$

But in that case, taking into account condition (139), which will be made along with (150) for sufficiently large $n > \bar{n} > \bar{n}_1$, we for all such values n must have:

$$\left[\frac{1}{N} \sum_{i=1}^N |\Omega_n(x_{n,i})|^2 \right]^{\frac{1}{2}} > \mu \left(\frac{\lambda}{l} \right)^{\frac{1}{2}} > \mu \frac{\lambda}{l}, \quad (151)$$

that, on comparison with (147), and it completes proof by contradiction.

Observation. By using the very light/lung modification of the given here reasonings, it is possible to be convinced analogously of the uniform limitedness of the coefficients of the polynomial $\phi_{m,N}(x)$, and also of absolute limitedness from below value q_N at $N \rightarrow \infty$: $\lim_{N \rightarrow \infty} q_N > 0$.

§14. We can now nearer approach the resolution of the question concerning the establishment of estimate/evaluations for values $\Delta_{m,N}, \beta_{m,N}$ [(138)].

Examining preliminarily problems (135) and (136) on any fixed/recorded multitude E_N [(134)] and completely without taking into account the behavior of functions $v(x)$ and of polynomial $\phi(x)$ on the remaining part of the segment $[a, b]$, we we can consider value \wedge [(137)], on the basis of the relationship

$$(q_N)^m (1 + 2\alpha_{m,N})^m < N(q_N)^m, \quad (152)$$

escape/ensuing from the requirement

$$\delta_{m,N}[\phi_{m,N}] < \delta_{m,N}[\phi_m].$$

Logarithmizing (152) (after decrease) and introducing, analogous to (18) and (20), the value:

$$\frac{\log(1+2a_{m,N})}{2a_{m,N}} = \sigma_{m,N}, \text{ where } \inf_m \sigma_{m,N} = \sigma_N, \quad (153)$$

we will obtain:

$$a_{m,N} < \frac{1}{2\sigma_N} \frac{\log N}{m}, \quad (154)$$

and, which means, then it is more

$$a_{m,N} < \frac{1}{2\sigma_N} \frac{\log N}{m}. \quad (155)$$

FOOTNOTE 36. In the case $a_{m,N}=0$, or even $\sigma_N=0$, which also is not excluded by the general conditions, accepted in previous paragraph (at least, for too large values of N), we will set/assume $\sigma_{m,N}=1$, and (154) surely it will remain valid, if we with $\sigma_N=0$ count always $a_{m,N}=0$. ENDFOOTNOTE.

A number σ_N here surely is positive in view of limitedness $a_{m,N}$, which with fixed E_N and alternating/variable $m > 1$ it is the corollary of the uniform (relative to m) limitedness of the coefficients of the polynomial $\phi_{m,N}(x)$, resulting from the general theorems of §1. In view of the clearly adjustable by

relationship (155) infinite smallness of $a_{m,N}$ when $m \rightarrow \infty$ (which, however, and priori it had to in this case to occur on the strength of common/general/total theorem of §2) we come further for the large values of m to the obvious precise conclusions:

$$\lim_{m \rightarrow \infty} \sigma_{m,N} = 1, \quad \lim_{m \rightarrow \infty} (m a_{m,N}) < \frac{1}{2} \log N. \quad (156)$$

If we begin now to vary in (152) not only m , but also N , then we logically will arrive at the need for introducing to examination value

$$\inf_{m,N} \sigma_{m,N} = \sigma. \quad (157)$$

The having most essential value for the justifiability of further conclusion/derivations fact of the positivity of this value is establish/installated analogously to fact $\sigma_N > 0$, but no longer on the basis of theorem of §1, but on the basis of the theorem, demonstrated in the previous paragraph, and observation to it. On the strength of (154) we have directly

$$a_{m,N} < \frac{1}{2^m} \frac{\log N}{m}. \quad (158)$$

In this relationship of parameters m and N ($m > 1$, $N \geq m + 1$) they can be varied completely independently of each other.

Let us allow however, that, increasing m , we simultaneously change and N as certain function of m ³⁷, thus, that with $m \rightarrow \infty$ we have $\log N/m \rightarrow 0$.

FOOTNOTE 37. We make this assumption for simplicity. It would be possible to visualize in more common/general/total form that m and N are changed simultaneously depending on any third value, which plays the role of their common/general/total argument. ENDFOOTNOTE.

Then value $a_{m,N}$ turns out to be on the strength of (156) infinitesimal, and this it imply, according to (153), (154), the relationship:

$$\lim_{\frac{\log N}{m} \rightarrow 0} a_{m,N} = 1, \quad \lim_{\frac{\log N}{m} \rightarrow 0} \left(a_{m,N} \cdot \frac{\log N}{m} \right) < \frac{1}{2}. \quad (159)$$

Finally, before going further, we will refine somewhat introduced by us under §13 condition (139) about approximate- the even distribution of the points of set E_N on segment $[a, b]$ ($b - a = 1$) during the large values of N . Specifically,, considering numbers (134) arrange/located in ascending order and designating

$$\lambda_N = \max \{x_1 - a, x_2 - x_1, x_3 - x_2, \dots, b - x_N\}, \quad (160)$$

we let us assume made the following condition of the extremity of two values s, s' : ³⁰

$$s = \overline{\lim}_{N \rightarrow \infty} \left(\lambda_N \cdot \frac{l}{N} \right) < \sup_{N \geq n+1} \left(\lambda_N \cdot \frac{l}{N} \right) = s' < \infty. \quad (161)$$

FOOTNOTE ³⁰. It is not difficult to comprehend that always $s \geq 1$.
ENDFOOTNOTE.

It is easy to understand that this last/latter condition is certain intensification of condition (139): under condition (161) surely turns out to be that which was made condition (139), besides with the constant value of parameter x (independent of the selection of the particular segment $[\alpha, \beta]$ on $[a, b]$), for such it will be possible to take any fixed/recorded positive number less than $1/s$.

§15. Now we can be converted already directly to values (138), for the purpose of evaluating the uniform on all segment $[a, b]$ approach/approximation, reached by polynomial $\Phi_{n,N}(x)$ - by the solution of the problem of the average exponential

approach/approximation on the final point set E_N (134) the same segment, comparing this polynomial with polynomial $\phi_0(x)$ [(12)], that represent the best uniform approximation (smallest uniform deviation from zero) on segment $[a, b]$.

Page 30.

With designations (10) - (13) and (135) - (138) we have first of all:

$$\Delta_{m,N} = \delta_0[\phi_{m,N}] - \epsilon \leq \{\delta_0[\phi_{m,N}] - \delta_{0N}[\phi_{m,N}]\} + \{\delta_{0N}[\phi_{m,N}] - \epsilon_N\}. \quad (162)$$

Assuming functions $b_i(x)$ ($i = 0, 1, \dots, n$) belonging to certain class, characterized by conditions of the type (61) with specific majorant function $\bar{\omega}(\delta)$, polynomials $\phi_{m,N}(x)$ on the strength of the demonstrated uniform limitedness of their coefficients will everything satisfy the same condition (61) with certain constant value of coefficient of K , depending neither on m nor of N , since we thus far leave aside the question concerning the refinement of the value of this coefficient. In that case we from (162) is easily concluded:

$$\Delta_{m,N} \leq K\bar{\omega}(\lambda_N) + 2u_{m,N}\epsilon_N \leq K\bar{\omega}\left(s_N \frac{1}{N}\right) + \frac{\epsilon \log N}{2m}, \quad (163)$$

where value $s_N = \lambda_N \cdot \frac{1}{N}$ it is bounded from two sides:

$$\frac{2}{3} < \frac{N}{N+1} < s_N = \lambda_N \cdot \frac{1}{N} < s'. \quad (164)$$

Let us introduce into more specific type examination the interpolational process, in which the selection of the final totally disconnected set E_N (134), with increasing m , is subordinated, besides condition (161), to the also following condition, which establish/install the general nature of dependence between parameters m and N :

$$\lambda_N \equiv s_N \frac{l}{N} = x[\varphi(qm)]. \quad (165)$$

Here, with the preservation/retention/maintaining of designations of §§9 and 10, $q=q_m$ it designates certain remaining at our disposal (for providing a larger flexibility of algorithm) positive parameter, which we at first, for a definition, also will assume bilaterally limited [with subsequent replacement of this requirement by more general conditions (172)]:

$$0 < c_1 < q < c_2. \quad (166)$$

From condition (165) we obtain, taking into account (93) and (95):

$$\log \frac{N}{s_N l} = \log \frac{1}{x[\varphi(qm)]} = qm \varphi(qm), \quad (167)$$

that directly it gives to us, by the way, convenient for many cases explicit expression for N in the form

$$N = s_N l e^{qm \varphi(qm)}, \quad (168)$$

showing that with $n \rightarrow \infty$ number N will grow always slower, in any case, than e^n .

Page 31.

On the other hand, then (167), taking into account (166), we have, obviously,

$$\log N = (1+\eta) \log \frac{1}{x[\varphi(qm)]} = (1+\eta) qm \varphi(qm) \quad (167')$$

($\eta \rightarrow 0$ with $m \rightarrow \infty$).

On the basis (165) and (167') relationship (163) assumes the form:

$$\Delta_{m,N} < K \varphi(qm) + \frac{(1+\eta)}{\underline{e}} qm \varphi(qm). \quad (169)$$

However, we can be freed from denominator \underline{e} in the second member in the form of the established/installed now infinite smallness of relation $\frac{\log N}{m} = (1+\eta) \varphi(qm)$; being returned to (163) and using now inequality (154), instead of (158), taking into account (159), we will assume $(1+\eta):e_{m,N} = 1+\epsilon$, that it will give to us:

$$\Delta_{m,N} < K \varphi(qm) + qm (1+\epsilon) \varphi(qm) \quad (\lim_{m \rightarrow \infty} \epsilon = 0) \quad (170)$$

or, dividing on $2q$ and remembering that $\Delta_{m,N} = q \cdot 2\beta_{m,N}$,

$$\beta_{m,N} < \frac{K}{2q} \varphi(qm) + \frac{1+\varepsilon}{2} q \varphi(qm) \quad (\varepsilon = \varepsilon_m \rightarrow 0 \text{ with } m \rightarrow \infty). \quad (171)$$

This means that taking account (96) that

$$\beta_{m,N} < \left(\frac{K}{2q} + \frac{1+\varepsilon}{2} q \right) q^{-\theta} \varphi(m) \quad (0 < \theta = \theta_m < 1). \quad (171')$$

Value $\beta_{m,N}$ characterizes, according to (138), the uniform on segment $[a, b]$ approach/approximation, reached by the interpolational polynomial of the average exponential approach/approximation $\phi_{m,N}(x)$. Equate/comparing obtained for it estimate/evaluation (171) - (171') with estimate/evaluation (104) - (104') - (121) - (123) analogous value α_m for the polynomial of the average exponential approach/approximation on the entire segment $\phi_m(x)$, taking into account the making more precise observations in §11, we first of all we can state/establish that significant circumstance that the replacement $\phi_m(x)$ by interpolational polynomial $\phi_{m,N}(x)$, with the observance of conditions (165) - (166) and (161) not only does not break the very fact of convergence to Chebyshev approach/approximation in the sense of the tendency toward zero of both differences

$$\delta_0[\phi_m] - \varepsilon \text{ and } \delta_0[\phi_{m,N}] - \varepsilon$$

but also leaves (in the general case) by constant the order of magnitude of the difference in question, which in both problems is

determined by the order of magnitude of function $\phi(m)$.

Page 32.

Being converted further to the examination of the coefficients with $\phi(m)$ in the compared estimates, we let us note, in the first place, that when Chebyshev problem (10) has infinite solution set whose coefficients can in that case be treated as coordinates of point $(c_m, c_{m+1}, \dots, c_{m+N})$ that which pass certain limited, closed, and convex totally disconnected set H_0 in the appropriate N-dimensional euclidean space ³⁹, about coefficient of K in (171) is represented possible to only say that to it can be assigned (at the large values of m) value, how conveniently close to the face side of the same structural coefficient K in relationship (61) for the mentioned set (H_0) of Chebyshev polynomials $\{\phi_m(x)\}$ ⁴⁰.

FOOTNOTES ³⁹. Comp. chapter VIII my, cited monograph [10].

⁴⁰. It is not difficult to see that the point $(c_{m,N+1}, c_{m,N+2}, \dots, c_{m,N+N})$

in the mentioned N-dimensional euclidean space infinite approaches, in the course of interpolation process in question, totally disconnected set H_0 . On the other hand, it is easy to comprehend that the order of the smallness of the distance of the

point indicated from set H_0 cannot be higher than the order of the smallness of values $\delta_{n,n}$ and $\beta_{n,n}$ consequently in the general case there cannot be higher than the order of the smallness of value $\delta(n)$ at $n \rightarrow \infty$. ENDFOOTNOTES.

But when solution $\phi_0(x)$ Chebyshev problem (10) turns out to be only, and specifically, consequently, in all cases when functions $\phi_1(x), \dots, \phi_n(x)$ compose on segment $[a, b]$ the set of functions of Chebyshev ¹¹ [in other words - when is satisfied the Haar condition [11], mentioned in footnote ⁵ §2], it will be possible to count coefficient of K (171) how convenient to close to the value of the mentioned structural coefficient of K for this only Chebyshev polynomial $\phi_0(x)$.

FOOTNOTE ¹¹ S. N. Bernshteyn, [21], Chapter 1. ENDFOOTNOTE.

In these cases we can identify coefficient of K in (171) with coefficient of k in (104) - (121), also, on this basis/base make a more precise quantitative comparison of estimate/evaluations (171) and (104) - (121). Relationship (171) can in actuality serve as starting point for different posing of the question concerning the

most advantageous possible selection of the value of the parameter $q=q_n$ in the determined by us interpolational process. We here will pause, mainly, at one of them, examining question from the standpoint of the accessible here maximum refinement of interpolational algorithm and planned more precise comparison of convergence properties of both processes with $n \rightarrow \infty$.

In the individualized study of problem for the different concrete/specific/actual classes of functions, we sometimes will have the capability, always retaining condition (161) and (165), substantial to widen condition (166), by set/assuming $c_1 = 0$ or by substituting c_2 on $+\infty$, by requiring only satisfaction of the conditions

$$\left. \begin{aligned} \varphi(qm) &= O[\varphi(m)] \\ \varphi\varphi(qm) &= O[\varphi(m)] \end{aligned} \right\} (q=q_n, m \rightarrow \infty). \quad (172)$$

Page 33.

With the observance of the first of these conditions (172) we surely will have $qm \rightarrow \infty$, and this - only, that is required for provision (167') and all remaining relationships from (167) to (171) inclusively also, as easily seen, they will remain completely in force under the conditions (172).

1. Case of functions $\varphi(x)$ satisfying the Lipshitz condition

of order r ($0 < r \leq 1$).

On the strength of (124) we have here under condition (166) ²:

$$\varphi(qm) = \frac{1+\varepsilon'}{q} \varphi(m); \quad (173)$$

$$\beta_{m,N} < \frac{K}{2q} \frac{1+\varepsilon'}{q} \varphi(m) + \frac{1+\varepsilon''}{2} \varphi(m). \quad (174)$$

FOOTNOTE ². By $\varepsilon', \varepsilon'', \varepsilon_1, \varepsilon_2$ and, etc subsequently, without specifying this, we understand some functions of m , which approach 0 themselves when $m \rightarrow \infty$. ENDFOOTNOTE.

Taking ε_1 in (166) by sufficiently large, we we can here do the first member is how many by conveniently small as compared with the second ³.

FOOTNOTE ³. Upon another posing of the question, without attaining (asymptotic) the minimization of expression (174 in question) [which in this case is reached by the price of an increase in the number of

used abscissas x_N in (135)], but desiring only as far as possible to smooth the values of both members in right side (174), we would be taken $\varphi = \frac{h}{\theta}$.

Let us note still that the ascending order in the number of interpolational abscissas x_N pri, $m \rightarrow \infty$ for the taken apart case more convenient anything is obtained not from (168), but directly to (165) in the form:

$$N = O\left[\left(\frac{m}{\log m}\right)^{\frac{1}{\delta}}\right]. \quad (175)$$

ENDFOOTNOTE.

Consequently, by designating through δ as small as desired, assigned on arbitrariness positive number, let us have, by taking into account again (124):

$$\beta_{m,N} < \frac{1+\delta}{2} \cdot \frac{1 \log m}{\varepsilon m} \quad (m > m_0), \quad (176)$$

and we see directly that the obtained upper boundary for $\beta_{m,N}$ asymptotically differs how conveniently little from the appropriate boundary for α_m that adjustable in this case by most precise relationship (27').

Finally, substituting condition (166) by less constrained

conditions (172) and taking, for example, it is concrete/specific/actual

$$q = q' \log m \quad (0 < c_1 < q' = q'_m < c_2) \quad (177)$$

where, as easily seen, (173) and (174) they preserve their form, we is already completely reached the asymptotic equality of both compared boundaries, since arbitrarily the close to unity coefficient $(1+\theta)$ in (176) is substituted by coefficient by infinitely close to unity.

FOOTNOTE **. In this case in relation (175) it is necessary to endow by divider/denominator $\log m$ but with a simultaneous improvement in the numerical coefficient. ENDFOOTNOTE.

Page 34.

2. Case of functions $v(x)$, satisfying the weakened condition D in 1st order (52).

Here we have accurately (comp. §11):

$$\varphi(M) = \frac{1}{\sqrt{M}}, \quad \varphi(qm) = q^{-\frac{1}{2}} \varphi(m). \quad (178)$$

Thus,

$$\beta_{n,N} < \frac{k}{2q} q^{-\frac{1}{2}} \varphi(m) + \frac{1+\varepsilon}{2} q^{\frac{1}{2}} \varphi(m). \quad (179)$$

With $q = \frac{k}{\varepsilon}$ we obtain:

$$\begin{aligned} \beta_{n,N} &< \frac{1}{2} \sqrt{\frac{k}{\varepsilon}} \varphi(m) + \frac{1+\varepsilon}{2} \sqrt{\frac{k}{\varepsilon}} \varphi(m) = \\ &= \left(1 + \frac{\varepsilon}{2}\right) \sqrt{\frac{k}{\varepsilon}} \varphi(m) = \left(1 + \frac{\varepsilon}{2}\right) \sqrt{\frac{k}{\varepsilon}} \cdot \frac{1}{\sqrt{m}}, \end{aligned} \quad (180)$$

and we again can accurately state/establish asymptotic agreement with upper bound for α_n adjustable by relationship (59).

3. The case of functions $v(x)$ satisfying the weakened condition Dinis (127) arbitrary order $\nu > 2$

Here, as can easily be seen from (128), accepting at first condition (166), we have:

$$\varphi(qm) = (1 + \varepsilon_1) \varphi(m). \quad (181)$$

Consequently,

$$\beta_{n,N} < \frac{k}{2q} (1 + \varepsilon_1) \varphi(m) + \frac{1 + \varepsilon_2}{2} q \varphi(m). \quad (182)$$

Taking now in (166) c_2 by sufficiently small let us find, it is analogous with that, as this was done in the first case:

$$\beta_{m,N} < \frac{k}{2q} (1+\delta) \cdot \frac{1}{\log^{(v-1)} m} \quad (m > m'), \quad (183)$$

understanding by δ as small as desired arbitrarily assigned (fix/recorded) positive number. We have, thus, as in the case 1°, asymptotically how conveniently close agreement of the specific here upper boundary for $\beta_{m,N}$ and with the most precise value of upper bound for α_m according to results of §11.

Page 35.

But also in this case accessible the full/total/complete asymptotic agreement (asymptotic equality) of both compared boundaries and besides with an advantageous decrease in number N of interpolational abscissas, if we, substituting condition (166) for (172), take concrete/specific/actually at least

$$q = \frac{q''}{\log m} \quad (0 < c_1 < q'' = q'_m < c_2) \quad (184)$$

with any $v > 2$ *5.

FOOTNOTE *5. In the relation to an asymptotic decrease in the number of interpolational abscissas here it would be possible to go and

considerably further, reinforcing speed of the decrease of the parameter $\varphi = \varphi_n$ at $n \rightarrow \infty$ (especially for $\nu > 2$) - either with observance (181), or with the replacement of the requirement for asymptotic equality (181) by certain inequality, let us say,

$$\lim_{n \rightarrow \infty} \frac{\varphi(\varphi n)}{\varphi(n)} < 2, \quad (181')$$

which, it goes without saying, will make it possible to achieve further decrease in the ascending order N and, etc. ENDFOOTNOTE.

The aggregate of the results, to which led us in the present work research presented, has nearest value for the questions of application/appendix to the Chebyshev problems proposed by me (see introduction) the method of the approximate construction of the solutions of the problems of the mean-exponential approximation, justifying, in particular, the application/use of an interpolational version of the mentioned method with the elimination of those difficulties of fundamental nature, into which runs the use in of this kind the questions of some commonly used in other cases of the methods of the approximation calculus of integrals.

Submitted 22.VIII 1947.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER FTD-ID(RS)T1265-77	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN THE AVERAGE EXPONENTIAL AND CHEBYSHEV APPROXIMATIONS (REPORT II)		5. TYPE OF REPORT & PERIOD COVERED Translation
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ye. Ya. Remez		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Foreign Technology Division Air Force Systems Command U. S. Air Force		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE 1948
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) 12		

DISTRIBUTION LIST

DISTRIBUTION DIRECT TO RECIPIENT

ORGANIZATION	MICROFICHE	ORGANIZATION	MICROFICHE
A205 DMATC	1	E053 AF/INAKA	1
A210 DMAAC	2	E017 AF/ RDXTR-W	1
B344 DIA/RDS-3C	8	E404 AEDC	1
C043 USAMIIA	1	E408 AFWL	1
C509 BALLISTIC RES LABS	1	E410 ADTC	1
C510 AIR MOBILITY R&D	1	E413 ESD	2
LAB/FIO		FTD	
C513 PICATINNY ARSENAL	1	CCN	1
C535 AVIATION SYS COMD	1	ETID	3
C557 USAIIC	1	NIA/PHS	1
C591 FSTC	5	NICD	5
C619 MIA REDSTONE	1		
D008 NISC	1		
H300 USAICE (USAREUR)	1		
P005 ERDA	1		
P055 CIA/CRS/ADD/SD	1		
NAVORDSTA (50L)	1		
NAVWPNSCEN (Code 121)	1		
NASA/KSI	1		
AFIT/LD	1		